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Minimization of the Constant in Inequalities of Jackson-Stechkin Type and the Value of Widths of Functional Classes in L2

Gulzorkhon Amirshoevich Yusupov*

Tajik State Pedagogical University by name S.Aini, Rudaky Avenue 121, 734003 Dushanbe, Tajikistan

***Corresponding Author**

Gulzorkhon Amirshoevich Yusupov, Tajik State Pedagogical University by name S.Aini, Rudaky avenue 121, 734003 Dushanbe, Tajikistan, Tel: 935002214, E-mail: G_7777@mail.ru

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Abstract

In this paper, we consider the problem of finding exact inequalities of Jackson-Stechkin type that are obtained for the average moduli of continuity of mth order ($m \in N$), with general weight function in L2 and also present applications. The exact values of these n-widths are calculated.

Keywords: The Space of Lebesgue; Trigonometric Polynomials; Weight Function; The Best of Approximation; Inequality; N-Widths

Introduction

Let L_2 := L2[0, 2π] denote the space of Lebesgue measurable 2π-periodic real functions f with norm

$$\|f\|_2 := \|f\|_{L_2} = \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2} < \infty.$$

Let ℑn−1 be the subspace of all trigonometric polynomials of degree ≤ n − 1. It is well known that, for any function f ∈ L_2 with Fourier expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\,,$$

the value of its best approximation in L2 by elements of the subspace ℑn−1 is

$$E_n(f) := \inf\left\{\|f - T_{n-1}\|_2 : \; T_{n-1}(x) \in \Im_{n-1}\right\}$$
$$= \|f - S_{n-1}(f)\|_2 = \left\{\sum_{k=n}^{\infty} \rho_k^2\right\}^{1/2}, \qquad (1.1)$$

where,

$$S_{n-1}(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx)$$

is the partial sum of order n − 1 of the Fourier series for the function f and $\rho_k^2 := a_k^2 + b_k^2.$

Let $\triangle^m_h$ (f)2 denote the norm of the mth-order difference of a function f ∈ L_2 with step h, that is,

$$\triangle_h^m(f)_2 := \|\triangle_h^m(f)\|_2 = \left\{\frac{1}{\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m}(-1)^k\binom{m}{k}f(x+kh)\right|^2 dx\right\}^{1/2}$$

Then

$$\omega_m(f;t) := \sup\left\{\triangle_m(f;h) : \; |h| \le t\right\}$$

defines the mth-order modulus of continuity of a function f ∈ L_2.

By $L_2^{(r)}$ (r ∈ ℕ; $L^{(0)}_2 = L_2$) we denote the set of functions f ∈ L_2, whose (r − 1)st derivatives are absolutely continuous and $f^{(r)}$ ∈ L_2. In Section 3, in defining classes of functions, we characterize the structural properties of a function f ∈ $L^{(r)}_2$ by the rate of convergence to zero of the modulus of continuity of its rth derivative $f^{(r)}$, defining this rate in terms of the majorant of some averaged quantity $\omega_m(f^{(r)}; t)$.

Related Extremal Problems

Extremal problems in the theory of approximation of differentiable periodic functions by trigonometric polynomials in the L_2 space involve the determination of sharp constants in inequalities of Jackson-Stechkin type

$$E_n(f) \le \chi n^{-r}\omega_m(f^{(r)}, t/n), \quad t > 0.$$

To this end, different extremal characteristics refining upper bounds for the constants χ were studied (see, for example, [1-8]).

Chernykh [1, pp.515-516] in studying the question of best approximation of differentiable periodic functions by trigonometric polynomials in L_2, showed that the functional

$$\Phi_n(f) := \left\{(n/2)\int_0^{\pi/n} \omega_m^2(f,t)\sin nt dt\right\}^{1/2}$$

is smaller than the Jackson functional ω_m(f, π/n) and, is apparently more natural for characterizing best approximations E_{n-1}(f) of periodic functions in L_2.

Given these considerations Ligun [2] studied extremal characteristics of the form (in what follows the ratio 0/0 is set equal to zero):

$$\mathcal{K}_{m,n,r}(\varphi,h) := \sup\left\{\frac{E_{n-1}^2(f)}{\int_0^h \omega_m^2(f^{(r)},t)\varphi(t)dt} : \; f \in L_2^{(r)}, f^{(r)} \ne const\right\},$$

where m, n ∈ N; r ∈ Z+; 0 < h 6 π/n; φ(t) > 0 is integrable on the segment [0, h]. He showed that

$$\left\{B_{n,h}^{r,m}(\varphi)\right\}^{-1} \leqslant \mathcal{K}_{m,n,r}(\varphi,h) \leqslant \left\{\inf_{n\leqslant k<\infty} B_{k,h}^{r,m}(\varphi)\right\}^{-1},$$

where

$$B_{k,h}^{r,m}(\varphi) := 2^m k^{2r}\int_0^h (1-\cos kt)^m \varphi(t)dt, \; k \geqslant n.$$

In order to generalize the results of [2], using the scheme of reasoning in Pinkus [3, pp.104-107], Shabozov and Yusupov [4] introduced the extremal characteristic

$$\chi_{m,n,r,p}(\varphi,h) := \sup\left\{\frac{E_{n-1}(f)}{\left(\int_0^h \omega_m^p(f^{(r)},t)\varphi(t)dt\right)^{1/p}} : f\in L_2^{(r)}, f^{(r)}\neq const\right\}, \quad (2.1)$$

where m, n ∈ N; r ∈ Z+; 0 < h 6 π/n; φ(t) > 0 is integrable on the segment [0, h], and for 0 < p 6 2 proved the inequality

$$\left\{\mathcal{A}_{n,h,p}^{r,m}(\varphi)\right\}^{-1} \leqslant \chi_{m,n,r,p}(\varphi,h) \leqslant \left\{\inf_{n\leqslant k<\infty}\mathcal{A}_{k,h,p}^{r,m}(\varphi)\right\}^{-1}, \quad (2.2)$$

where

$$\mathcal{A}_{k,h,p}^{r,m}(\varphi) := 2^{m/2}\left(k^{rp}\int_0^h (1-\cos kt)^{mp/2}\varphi(t)dt\right)^{1/p}, \quad k \geqslant n.$$

In the calculation of the exact values of the n-widths of classes of functions directly from (2.1), and in connection with the accuracy of (2.2) there is a need to establish the equality

$$\inf_{n\leq k<\infty} \mathcal{A}_{k,h,p}^{r,m}(\varphi) = \mathcal{A}_{n,h,p}^{r,m}(\varphi), \quad (2.3)$$

for any positive integrable functions φ on the segment [0, h]. In general, the verification of (2.3) is not always possible. For some specific weight functions φ, condition (2.3) is proved in [5]. Obviously, equation (2.3) depends on the structural properties of the weight function φ. A natural question arises: what structural and differential properties must a function φ have in order to satisfy (2.3)? The answer to the question is contained in the following statement.

Theorem 2.1. Suppose that the weight function φ (t) defined on [0, h] is non-negative and continuously differentiable thereon. If, for r ∈ ℕ, 1/r < p ≤ 2 and every t ∈ [0, h] we have

$$(rp - 1)\varphi(t) - t\varphi'(t) \geq 0, \quad (2.4)$$

then, for any m, n ∈ ℕ and 0 < h ≤ π/n we have the equality

$$\chi_{m,n,r,p}(\varphi;h) = 2^{-m}n^{-r}\left(\int_0^h\left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt\right)^{-1/p}. \quad (2.5)$$

There is a function $f_0 \in L_2^{(r)}$, $f_0^{(r)} \neq$const, realizing the upper bound in (2.1) equal to (2.5).

Proof. We use the following simplified version of Minkowski's inequality [3, p.104]

$$\left(\int_0^h\left(\sum_{k=n}^{\infty}|f_k(t)|^2\right)^{p/2}\varphi(t)dt\right)^{1/p} \geq \left(\sum_{k=n}^{\infty}\left(\int_0^h |f_k(t)|^p\varphi(t)dt\right)^{2/p}\right)^{1/2},$$

$$(0<p\leq 2;\ \varphi(t)\geq 0,\ 0<t\leq h).$$

Indeed, bearing in mind that for any function $f \in L_2^{(r)}$ we have the relation [10]

$$\omega_m^2(f^{(r)};t) = 2^m \sup\left\{\sum_{k=1}^{\infty} k^{2r}\rho_k^2(1-\cos ku)^m : |u|\leq t\right\},$$

then

$$\left(\int_0^h \omega_m^p(f^{(r)};t)\varphi(t)dt\right)^{1/p}$$
$$\geq \left\{\int_0^h\left[2^m\sum_{k=n}^{\infty}k^{2r}\rho_k^2\left(1-\cos kt\right)^m\right]^{p/2}\varphi(t)dt\right\}^{1/p}$$
$$= \left\{\int_0^h\left(2^m\sum_{k=n}^{\infty}k^{2r}\rho_k^2(1-\cos kt)^m[\varphi(t)]^{2/p}\right)^{p/2}dt\right\}^{1/p}$$
$$\geq \left\{2^m\sum_{k=n}^{\infty}\rho_k^2\left(k^{rp}\int_0^h(1-\cos kt)^{mp/2}\varphi(t)dt\right)^{2/p}\right\}^{1/2}. \quad (2.6)$$

We prove that under our assumptions

$$y(x) = x^{rp}\int_0^h(1-\cos xt)^{mp/2}\varphi(t)dt \quad (2.7)$$

is a strictly increasing function in the domain Q = { x : x ≥ 0 } and, hence,

$$\min\left\{y(x):\ x\in Q\right\} = y(n) := n^{rp}\int_0^h(1-\cos nt)^{mp/2}\varphi(t)dt. \quad (2.8)$$

Indeed, differentiating (2.7) and using the elementary identity

$$\frac{d}{dx}(1-\cos xt)^{mp/2} = \frac{t}{x}\cdot\frac{d}{dt}(1-\cos xt)^{mp/2},$$

we have

$$y'(x) = rpx^{rp-1}\int_0^h(1-\cos xt)^{mp/2}\varphi(t)dt + x^{rp}\int_0^h\frac{d}{dx}(1-\cos xt)^{mp/2}\varphi(t)dt$$

$$= rpx^{rp-1}\int_0^h(1-\cos xt)^{mp/2}\varphi(t)dt + x^{rp-1}\int_0^h\frac{d}{dt}(1-\cos xt)^{mp/2}t\varphi(t)dt. \quad (2.9)$$

Integrating by parts in the last integral of (2.9), the inequality (2.4), we finally obtain

$$y'(x) = x^{rp-1}\left\{ (1-\cos hx)^{mp/2} h\varphi(h) + \int_0^h (1-\cos xt)^{mp/2}\left[(rp-1)\varphi(t) - t\varphi'(t)\right] dt \right\} \geq 0,$$

which implies the relation (2.8).

Therefore continuing inequality (2.6), we have

$$\ldots \geq 2^{m/2} n^r \left(\int_0^h (1-\cos nt)^{mp/2}\varphi(t)dt\right)^{1/p}\left\{\sum_{k=n}^{\infty}\rho_k^2\right\}^{1/2} = 2^m n^r E_{n-1}(f)\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt\right)^{1/p}. \quad (2.10)$$

From (2.6) and (2.10) it follows that

$$\frac{E_{n-1}(f)}{\left(\int_0^h \omega_m^p(f^{(r)};t)\varphi(t)dt\right)^{1/p}} \leq 2^{-m}n^{-r}\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt\right)^{-1/p}.$$

Since the last inequality holds for any f ∈ L (r) 2 we have an upper bound for (2.5):

$$\chi_{m,n,r,p}(\varphi;h) \leq 2^{-m}n^{-r}\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt\right)^{-1/p}. \quad (2.11)$$

The lower bound in (2.5), valid for all $0 < h \leq \pi/n$, is obtained by using the function $f_0(x) = \cos nx \in L^{(r)}_2$. We have

$$E_{n-1}(f_0) = 1, \quad \omega_m(f_0^{(r)};t) = 2^m n^r \left(\sin\frac{nt}{2}\right)^m, \quad 0 < nt \leq \pi.$$

Thus

$$\chi_{m,n,r,p}(\varphi;h) \geq \frac{E_{n-1}(f_0)}{\left(\int_0^h \omega_m^p(f_0^{(r)};t)\varphi(t)dt\right)^{1/p}} = 2^{-m}n^{-r}\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt\right)^{-1/p}. \quad (2.13)$$

Equation (2.5) is a consequence of (2.11) and (2.12). This completes the proof of Theorem 2.1.

As a particular case of Theorem 2.1 we have:

Corollary 2.1. *Let* $\varphi(t) = \sin^\gamma\left(\frac{\beta t}{h}\right), 0 < \beta \leq \pi, 0 \leq t \leq h, 0 < h \leq \pi/n, 0 \leq \gamma \leq rp-1, 1/r < p \leq 2, r \in \mathbb{N}$. *Then for any* $m, n \in \mathbb{N}$ *we have*

$$\chi_{m,n,r,p}\left(\sin^\gamma\left(\frac{\beta t}{h}\right);h\right) = 2^{-m}n^{-r}\left\{\int_0^h \left(\sin\left(\frac{nt}{2}\right)\right)^{mp}\sin^\gamma\left(\frac{\beta t}{h}\right)dt\right\}^{-1/p}. \quad (2.13)$$

Proof. The parameter values p, r, β, γ, h as in the statement of Corollary 2.1 suffice to verify (2.4). We have

$$(rp-1)\varphi(t) - t\varphi'(t) = (rp-1)\sin^\gamma\left(\frac{\beta t}{h}\right) - t\gamma\frac{\beta}{h}\sin^{\gamma-1}\left(\frac{\beta t}{h}\right)\cos\frac{\beta t}{h}$$

$$= \left[(rp-1) - \gamma\frac{\beta t}{h}\operatorname{ctg}\frac{\beta t}{h}\right]\sin^\gamma\left(\frac{\beta t}{h}\right) \geq \left[(rp-1)-\gamma\right]\sin^\gamma\left(\frac{\beta t}{h}\right) \geq 0,$$

because for the values of the above parameters

$$\min\left\{\left[(rp-1) - \gamma\cdot\frac{\beta t}{h}\operatorname{ctg}\frac{\beta t}{h}\right] : t \in [0,h]\right\} = (rp-1) - \gamma \geq 0.$$

This proves Corollary 2.1.

Corollary 2.1 contains, in particular, the results of [4-8] for different parameters p, γ, β and h.

The Statement of the Main Results

We recall the necessary concepts and definitions which will be used later. Suppose that $S = \{v : \|v\| \leq 1\}$ is the unit ball in L_2; $\mathfrak{M}$ is a convex centrally symmetric subset from L_2; $\Lambda_N \subset L_2$ is an N-dimensional subspace; $\Lambda^N \subset L_2$ is a subspace of codimension N; $L : L_2 \to \Lambda_N$ is a continuous linear operator taking elements of the space L_2 to Λ_N; and $L^\perp : L_2 \to \Lambda_N$ is a continuous linear projection operator from L_2 onto Λ_N. The quantities

$$b_N(\mathfrak{M}, L_2) = \sup\left\{\sup\{\varepsilon > 0;\ \varepsilon S \cap L_{N+1} \subset \mathfrak{M}\} : \Lambda_{N+1} \subset L_2\right\},$$

$$d^N(\mathfrak{M}, L_2) = \inf\{\sup\{\|f\|_2 : f \in \mathfrak{M} \cap \Lambda^N\} : \Lambda^N \subset L_2\},$$
$$d_N(\mathfrak{M}, L_2) = \inf\{\sup\{\inf\{\|f-g\|_2 : g \in \Lambda_N\} : f \in \mathfrak{M}\} : \Lambda_N \subset L_2\},$$
$$\lambda_N(\mathfrak{M}, L_2) = \inf\{\inf\{\sup\{\|f - \mathcal{L}f\|_2 : f \in \mathfrak{M}\} : \mathcal{L}L_2 \subset \Lambda_N\} : \Lambda_N \subset L_2\},$$
$$\pi_N(\mathfrak{M}, L_2) = \inf\{\inf\{\sup\{\|f - \mathcal{L}^{\perp}f\|_2 : f \in \mathfrak{M}\} : \mathcal{L}^{\perp}L_2 \subset \Lambda_N\} : \Lambda_N \subset L_2\}$$

are called, respectively, the Bernstein, Gelfand, Kolmogorov, linear, and projection N-widths in the space L_2. Since L_2 is a Hilbert space, the N-widths listed above are related by (see, e.g., [3]):

$$b_N(\mathfrak{M}; L_2) \le d^N(\mathfrak{M}; L_2) \le d_N(\mathfrak{M}; L_2) = \lambda_N(\mathfrak{M}; L_2) = \pi_N(\mathfrak{M}; L_2). \qquad (3.1)$$

We shall denote by $W_m(f^{(r)}; \varphi)p,h$, $m \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $0 < p \le 2$, $0 < h \le \pi$, the pth mean value of the modulus of continuity of mth order of functions $f^{(r)}$ with weight $\varphi(t)$:

$$W_m(f^{(r)};\varphi)_{p,h} = \left(\int_0^h \omega_m^p(f^{(r)};t)\varphi(t)dt\right)^{1/p}\left(\int_0^h \varphi(t)dt\right)^{-1/p}, \qquad (3.2)$$

and, by $L^{(r)}_2(m, p, h; \varphi)$ we designate the set of functions $f \in L^{(r)}_2$ for which $W_m(f^{(r)}; \varphi)p,h \le 1$.

Obviously, because of the monotonicity of the modulus of continuity of mth order $\omega_m(f^{(r)};t)$ for an arbitrary integrable weight function $\varphi(t) \ge 0$ $(0 < t \le h)$ from (3.2) it follows that

$C(m, r, p, h)\, \omega_m(f^{(r)}; h) \le Wm(f^{(r)}; \varphi)p,h \le \omega(f^{(r)}; h)$,

where C(m, r, p, h) is a positive constant that depends on the values of the parameters in parentheses. With this notation, the search for the smallest constant in the Jackson-Stechkin inequality is equivalent to the problem of computing the exact upper bound

$$\mathrm{K}_{m,p,h}\left(L_2^{(r)}, L_2, \Im_N\right) = \sup\left\{\frac{E(f,\Im_N)_2}{W_m(f^{(r)};\varphi)_{p,h}} : f \in L_2^{(r)}\right\}. \qquad (3.3)$$

Here we will look for the lowest constant relative to the entire set of the spaces $\Im N \subset L_2$ of fixed dimension N. This will show that the result can not be improved upon by switching to another subspace of the same dimension

$$\mathrm{K}_{N,m,p,h}\left(L_2^{(r)}, L_2\right) = \inf\left\{\mathrm{K}_{m,p,h}\left(L_2^{(r)}, L_2, \Im_N\right) : \Im_N \subset L_2\right\}$$
$$= \inf\left\{\sup\left\{\frac{E(f,\Im_N)_2}{W_m(f^{(r)};\varphi)_{p,h}} : f \in L_2^{(r)}\right\} : \Im_N \subset L_2\right\}.$$

We also put

$$E_{n-1}\left(L_2^{(r)}(m,p,h;\varphi)\right)_2 = \sup\left\{\|f - S_{n-1}(f)\|_2 : f \in L_2^{(r)}(m,p,h;\varphi)\right\}.$$

Proposition 3.1. Suppose that h, p > 0, $r \in \mathbb{Z}_{+}$, m, $n \in \mathbb{N}$. Then the following inequality holds

$K_{N,m,p,h}(L_2^{(r)}, L_2) = d_N(L_2^{(r)}(m, p, h; \varphi), L_2)$.

Proof. If $f \in L^{(r)}_2$, $\varphi(t) \ge 0$ is integrable on the segment [0, h] and, $W_m(f^{(r)}, \varphi)p,h = \alpha > 0$ then for $f1(x) = \alpha^{-1} f(x)$, we have $Wm(f^{(r)}_1, \varphi)p,h = 1$. Given the positive homogeneous functional $E(f, \Im N)2$ and $W_m(f^{(r)}, \varphi)p,h$, for any $0 < p \le 2$ and a fixed $h > 0$ we have

$$\sup_{f\in L_2^{(r)}} \frac{E(f,\Im_N)_2}{W_m(f^{(r)};\varphi)_{p,h}} \le \sup_{f\in L_2^{(r)}(m,p,h;\varphi)} E(f,\Im_N)_2. \qquad (3.4)$$

Through (3.4) the lower bound over all subspaces $\Im N \subset L_2$ dimension N we obtain

$K_{N,m,p,h}(L_2^{(r)}, L_2) = d_N(L_2^{(r)}(m, p, h; \varphi), L_2)$.

On the other hand, for any function $f \in L^{(r)}_2(m, p, h; \varphi)$ by definition of the class $L^{(r)}_2(m, p, h; \varphi)$ have an inequality of the form

$$E(f,\Im_N)_2 \le \frac{E(f,\Im_N)_2}{W_m(f^{(r)};\varphi)_{p,h}}$$

and as this is true for every subspace $\Im N \subset L_2$ then

$d_N(L_2^{(r)}(m, p, h; \varphi), L_2) \le K_{N,m,p,h}((L_2^{(r)}, (L_2)$.

Proposition 3.1 follows from (3.5) and (3.6).

Theorem 3.1. Suppose that the weight function $\varphi(t)$ defined on the segment [0, h] is non-negative and continuously differentiable thereon. If for some $r \in \mathbb{N}$, $1/r < p \le 2$, and any $t \in [0, h]$, we have

$(rp - 1)\,\varphi(t) - t\,\varphi'(t) \ge 0, \ldots$ (3.7)

then, for any m, $n \in \mathbb{N}$ and $0 < h \le \pi/n$

$K_{2n,m,p,h}(L_2^{(r)}, L_2) = K_{2n-1,m,p,h}(L_2^{(r)}, L_2) = E_{n-1}(L_2^{(r)}(m, p,h; \varphi)_2$

where $\delta_k(\cdot)$ are any of the k-widths: Bernstein $b_k(\cdot)$, Kolmogorov $d_k(\cdot)$, linear $\lambda_k(\cdot)$, Gelfand $d_k(\cdot)$ or projection $\pi_k(\cdot)$. All k–widths are attained by taking the partial sums of the Fourier series

$S_{k-1}(f;t)$.

Proof. From (2.10) and since (3.7) holds, it follows that

$$E_{n-1}(f) \le 2^{-m} n^{-r} \frac{\left(\int_0^h \omega_m^p(f^{(r)};t)\,\varphi(t)dt\right)^{1/p}}{\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \varphi(t)dt\right)^{1/p}}$$

$$= 2^{-m} n^{-r} \left(\frac{\int_0^h \omega_m^p(f^{(r)};t)\,\varphi(t)dt}{\int_0^h \varphi(t)dt}\right)^{1/p} \cdot \left(\frac{\int_0^h \varphi(t)dt}{\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \varphi(t)dt}\right)^{1/p}.$$

Hence, for the width of the projection of class $L^{(r)}_2(m, p, h; \varphi)$, we obtain an upper bound

$$\pi_{2n}\left(L_2^{(r)}(m,p,h;\varphi), L_2\right) \le \pi_{2n-1}\left(L_2^{(r)}(m,p,h;\varphi), L_2\right) \le E_{n-1}\left(L_2^{(r)}(m,p,h;\varphi)\right)_2$$

$$= 2^{-m} n^{-r} \left\{\int_0^h \varphi(t)dt\right\}^{1/p} \left\{\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \varphi(t)dt\right\}^{-1/p}. \quad (3.8)$$

In order to obtain a lower bound for the Bernstein n-width of $L^{(r)}_2(m, p, h; \varphi)$ we consider the ball

$$\mathbb{B}_{2n+1} = \left\{ T_n \in \Im_{2n+1} : \|T_n\|_2 \le \frac{2^{-m} n^{-r} \left(\int_0^h \varphi(t)dt\right)^{1/p}}{\left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \varphi(t)dt\right)^{1/p}} \right\}$$

in the (2n+1)–dimensional subspace $\Im 2_{n+1}$ of trigonometric polynomials and show that $\mathbb{B}_{2n+1} \subset L^{(r)}_2(m, p, h; \varphi)$. In [7] it is proved that for an arbitrary polynomial $T_n \in \Im_{2n+1}$ for $0 < h \le \pi/n$, the inequality

$$\omega_m\left(T_n^{(r)}, t\right) \le 2^m n^r \left(\sin\frac{nt}{2}\right)^m \|T_n\|_2. \quad (3.9)$$

Elevate both sides of (3.9) to the to the power p, multiply by φ and integrate the result over t in the range from 0 to h. Whence we obtain directly

$$\left(\frac{\int_0^h \omega_m^p\left(T_n^{(r)};t\right)\varphi(t)dt}{\int_0^h \varphi(t)dt}\right)^{1/p} \le 2^m n^r \left(\frac{\int_0^h \left(\sin\frac{nt}{2}\right)^{mp}\varphi(t)dt}{\int_0^h \varphi(t)dt}\right)^{1/p} \cdot \|T_n\|_2 \le 1,$$

from which $\mathbb{B}_{2n+1} \subset L^{(r)}_2(m, p, h; \varphi)$. By the definition of the Bernstein n-width we obtain the lower bound

$$b_{2n-1}\left(L_2^{(r)}(m,p,h;\varphi), L_2\right) \ge b_{2n}\left(L_2^{(r)}(m,p,h;\varphi), L_2\right) \ge b_{2n}\left(\mathbb{B}_{2n+1}, L_2\right)$$

$$= 2^{-m} n^{-r} \left\{\int_0^h \varphi(t)dt\right\}^{1/p} \left\{\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \varphi(t)dt\right\}^{-1/p}. \quad (3.10)$$

This completes the proof of Theorem 3.1.

As a particular case of Theorem 3.1 we have a result found in [5]. Namely,

Corollary 3.1. Let $Let\ \varphi_*(t) = \sin^\gamma \frac{\beta}{h}t;\ 0 \le \gamma \le rp-1, r \ge 1, 1/r < p \le 2.$

Then the following inequality holds

$$\mathbb{K}_{2n,m,p,h}\left(L_2^{(r)}, L_2\right) = \mathbb{K}_{2n-1,m,p,h}\left(L_2^{(r)}, L_2\right)$$

$$= \delta_{2n}\left(L_2^{(r)}(p,h,m;\varphi_*), L_2\right) = \delta_{2n-1}\left(L_2^{(r)}(p,h,m;\varphi_*), L_2\right)$$

$$= 2^{-m} n^{-r} \left(\int_0^h \sin^\gamma \frac{\beta}{h} t dt\right)^{1/p} \left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma \frac{\beta}{h} t dt\right)^{-1/p},$$

where $\delta_k(\cdot)$ are any of the above-listed k-widths

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